CLNS 96/1392

Large N_c Universality of The Baryon Isgur–Wise Form Factor: The Group Theoretical Approach

Chi-Keung Chow

Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, NY 14853.

(February 1, 2008)

Abstract

In a previous article, it has been proved under the framework of chiral soliton model that the same Isgur–Wise form factor describes the semileptonic $\Lambda_b \to \Lambda_c$ and $\Sigma_b^{(*)} \to \Sigma_c^{(*)}$ decays in the large N_c limit. It is shown here that this result is in fact independent of the chiral soliton model and is solely the consequence of the spin-flavor SU(4) symmetry which arises in the baryon sector in the large N_c limit.

In a previous article [1], it has been proved that the semileptonic $\Lambda_b \to \Lambda_c$ and $\Sigma_b^{(*)} \to \Sigma_c^{(*)}$ decays are controlled by the same Isgur–Wise form factor in the large N_c limit. Recall that the semileptonic $\Lambda_b \to \Lambda_c$ decay depends on a universal form factor $\eta(w)$ [2–6], which is defined by

$$\langle \Lambda_c(v',s')|\, \bar{c}\Gamma b\, |\Lambda_b(v,s)\rangle = \eta(w)\, \bar{u}_{\Lambda_c}(v',s')\,\Gamma\, u_{\Lambda_b}(v,s), \tag{1}$$

where $w = v \cdot v'$. For the semileptonic $\Sigma_b^{(*)} \to \Sigma_c^{(*)}$ decay we have two Isgur–Wise form factors, $\zeta_1(w)$ and $\zeta_2(w)$ [3–6].

$$\begin{aligned}&\langle \Sigma_c^{(*)}(v',s')|\, \bar{c}\Gamma b\, |\Sigma_b^{(*)}(v,s)\rangle \\ &= (\zeta_1(w) g_{\mu\nu} + \zeta_2(w) v_\nu v'_\mu)\, \bar{u}^\nu_{\Sigma_c^{(*)}}(v',s')\,\Gamma\, u^\mu_{\Sigma_b^{(*)}}(v,s),\end{aligned} \tag{2}$$

where $u^\nu_{\Sigma_b^*}(v',s')$ is the Rarita–Schwinger spinor vector for a spin-$\frac{3}{2}$ particle and $u^\mu_{\Sigma_b}(v,s)$ is defined by

$$u^\mu_{\Sigma_b}(v,s) = \frac{(\gamma^\mu + v^\mu)\gamma_5}{\sqrt{3}} u_{\Sigma_b}(v,s) \tag{3}$$

and similarly for $u^\nu_{\Sigma_c^{(*)}}(v',s')$. In Ref. [1], it has been shown that

$$\zeta_1(w) = -(1+w)\zeta_2 = \eta(w). \tag{4}$$

i.e., the same form factor describes both semileptonic transitions.

The large N_c limit was studied in Ref. [1] under the framework of the chiral soliton model, which is generally believed to be the realization of large N_c QCD in the baryon sector, though the equivalence has not yet been rigorously proved. Naturally it raises the question whether the same universality of baryon Isgur–Wise form factor in the large N_c limit can be obtained without reference to the chiral soliton model. In a recent paper [7] it was attempted to get constraints on the Isgur–Wise form factors from unitarity in 1-pion loop renormalization of the $\Lambda_b \to \Lambda_c$ decay and weak decays with 1 or 2 pion emission. Since pion-baryon Yukawa couplings are of order $N_c^{1/2}$, many individual weak decay graphs involving pion lines diverge in the large N_c limit. To preserve unitarity, non-trivial cancelation must take place between

graphs with intermediate Λ_Q and $\Sigma_Q^{(*)}$ states, and hence giving non-trivial relations between the Λ_Q and $\Sigma_Q^{(*)}$ Isgur–Wise form factors. Ref. [7] found that the relations obtained by this method are "consistent with, but not as powerful as" those obtained in Ref. [1] through the chiral soliton model. In particular, no constraint can be placed on $\zeta_2(w)$ in Ref. [7]. It is not surprising by noting that $\zeta_2(w)$ contributes only away from the point of zero recoil, where the Isgur–Wise form factors vanish like $\exp(-N_c^{3/2})$ in the large N_c limit [8], faster than N_c^{-n} for any finite positive n.

In this article, it will be attempted to reproduce relation (4) without using the chiral soliton model. We will use the formalism of large N_c baryon developed in Ref. [9], which depends on group theoretical considerations and is completely model independent. It is found that relation (4) can indeed be reproduced and hence the result of Ref. [1] follows solely from the large N_c limit without any additional assumptions.

We will first review the SU(4) spin-flavor symmetry for large N_c baryons developed in Ref. [9]. The symmetry is defined by the commutation relations,

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \quad [I^a, I^b] = i\epsilon^{abc} I^c, \quad [I^a, J^i] = 0, \tag{5a}$$

$$[J^i, X_0^{jb}] = i\epsilon^{ijk} X_0^{kb}, \quad [I^a, X_0^{jb}] = i\epsilon^{abc} X_0^{jc}, \tag{5b}$$

$$[X_0^{ia}, X_0^{jb}] = 0, \tag{5c}$$

where I^a and J^i are the isospin and spin operators respectively, and X_0^{ia} is the baryon axial current matrix element in leading order of the $1/N_c$ expansion, defined by

$$\langle B'|\bar{q}\gamma^i\gamma_5\tau^a q|B\rangle = N_c g (X_0^{ia})_{B'B} + \text{higher order in } 1/N_c. \tag{6}$$

Eq. (5a) is just the usual commutation relations for $\mathrm{SU}(2)_I \otimes \mathrm{SU}(2)_J$, while Eq. (5b) states that fact that the axial current couplings are of spin 1 and isospin 1. Lastly, Eq. (5c) follows from unitary constraint of pion-baryon scattering and is the starting point of the formalism developed in Ref. [9].

The induced representation of this spin-flavor SU(4) has been discussed in detail in Ref. [9] and will not be repeated here. A vector under the induced representation is an eigenvector of X_0^{ia} (we will denote the eigenvalue by X_0^{ia} as well) and also labeled by its transformation properties under the little group. For the case of physical interest, the little group is SU(2) $\times Z_2$, and the state would be denoted as $|X_0^{ia}, K, k, \pm\rangle$ where (K, k) and $\pm$ are the representations under the little groups SU(2) and Z_2 respectively. It can be shown that $\vec{K} = \vec{I} + \vec{J}$ and $Z_2 = +$ and $-$ for bosonic and fermionic states respectively. The prime example in Ref. [9] are the four nucleon states with $(I, J) = (\frac{1}{2}, \frac{1}{2})$ and the sixteen Delta states with $(I, J) = (\frac{3}{2}, \frac{3}{2})$ which fall under the **20** representation under the spin-flavor SU(4). These states can be constructed out of the basis $|X_0^{ia}, 0, 0, -\rangle$, as $\vec{K} = \vec{I} + \vec{J} = 0$ and both the nucleon and the Delta are fermions.

To apply this formalism to heavy quark states, a straightforward application will be to follow the treatment of hyperons in Ref. [9] and consider the induced representation with $\vec{K} = \vec{I} + \vec{J} = \frac{1}{2}$ and $Z_2 = -$. But it will be much more convenient and illuminating to study the induced representation describing just the "brown mucks" of the heavy baryons without the heavy quark. More exactly, instead of considering the spin-flavor SU(4) generated by (I^a, J^i, X_0^{ia}), one can consider instead that generated by $(I^a, s_\ell^i, X_0^{ia})$, with s_ℓ the spin of the "brown muck"[1]. The SU(4) commutation relations stay unchanged, and the "brown mucks" of Λ_Q and $\Sigma_Q^{(*)}$, with $(I, s_\ell) = (0, 0)$ and $(1, 1)$ respectively, form an SU(4) **10** representation. Now $\vec{K} = \vec{I} + \vec{s}_\ell = 0$ and $Z_2 = +$ as the "brown mucks" are bosonic for odd N_c. Hence the heavy quark "brown muck" states can be constructed out of $|X_0^{ia}, 0, 0, +\rangle$. Since we are not going to concern about the transformation properties under the little group for the rest of

[1]The spin of light degrees of freedom s_ℓ is well defined and conserved, a consequence of heavy quark symmetry. Recall that $\vec{J} = \vec{s}_Q + \vec{s}_\ell$. The conservation of s_ℓ follows from the conservation of the heavy quark spin s_Q in the heavy quark limit. A similar treatment in the hyperon sector will be problematic as the spin of the strange quark is not conserved.

our discussion, these state will be denoted simply as $|X_0^{ia}\rangle$ below.

It is in place to discuss the properties of the states $|X_0^{ia}\rangle$. Since a change of the normalization of X_0^{ia} is equivalent to a redefinition of the axial current coupling constant g in Eq. (6), we can, without loss of generality, impose the renormalization that

$$X_0^{ia} X_0^{ia} = \mathrm{Tr} X_0^2 = 3. \tag{7}$$

The states with different X_0^{ia} eigenvalues can be rotated or iso-rotated into each other.

$$U_{s_\ell}(g)|X_0^{ia}\rangle = |D^{ij}(g) X_0^{ja}\rangle, \tag{8a}$$

$$U_I(h)|X_0^{ia}\rangle = |D^{ab}(h) X_0^{ib}\rangle, \tag{8b}$$

where $U_{s_\ell}(g)$ is the unitary transformation corresponding to a finite spin rotation by the $g \in \mathrm{SU}(2)_{s_\ell}$, $D^{ij}(g)$ is the usual rotation matrix in 3-dimensions, while $U_I(h)$ and $D^{ab}(h)$ are the counterparts in isospace for $h \in \mathrm{SU}(2)_I$. Moreover, since $\vec{K} = \vec{I} + \vec{s}_\ell = 0$, for any state $|X_0^{ia}\rangle$ and any $g \in \mathrm{SU}(2)_{s_\ell}$, there exist a certain $h \in \mathrm{SU}(2)_I$ such that

$$U_{s_\ell}(g)|X_0^{ia}\rangle = U_I(h)|X_0^{ia}\rangle, \tag{9}$$

i.e., an isorotation is equivalent to a rotation. In particular, if we choose $X_0^{ia} = \overline{X}_0 \equiv \mathrm{diag}(1,1,1)$, Eq. (9) is satisfied by $g = h$.

This opens up the possibility of labeling the set of states $\{|X_0^{ia}\rangle : \mathrm{Tr} X_0^2 = 3\}$ by SU(2) elements. With the definition

$$|X_h\rangle = U_I(h)|\overline{X}_0\rangle, \tag{10}$$

the set $\{|X_h\rangle : h \in SU(2)_I\}$ are orthogonal.

$$\langle X_{h'}|X_h\rangle = \delta(h'h^{-1}), \tag{11}$$

where $\delta(g)$ is a δ-function on the SU(2) group normalized so that $\int dg \delta(g) = 1$. This association of the states to $\mathrm{SU}(2)_I$ elements is crucial to our proof, as will be shown below.

So far the heavy quark has not yet appeared in our discussion. In the heavy quark limit, the heavy quark is just the source of a static color field in which the "brown mucks" appear as eigenstates. During a $b \to c$ transition, all the "brown muck" feels is the change of the velocity of the color source. In this language, the Isgur–Wise form factors are just the overlap of the initial and final "brown mucks" [10]. Now, let the state $|\overline{X}_0\rangle$ discussed above be one moving with velocity v. Of all the normalized baryon "brown mucks" moving with velocity v', we will denote the one which overlaps maximally with $|\overline{X}_0\rangle$ as $|\overline{X}'_0\rangle$. Analogous to Eq. (10), we define

$$|X'_h\rangle = U_I(h)|\overline{X}'_0\rangle. \tag{12}$$

Then it is trivial to prove that

$$\langle X'_{h'}|\overline{X}_0\rangle \sim \delta(h'). \tag{13}$$

(To prove this, assume the contrary and there exist a certain non-trivial h' for which $\langle X'_{h'}|\overline{X}_0\rangle > 0$. Then some normalized linear combination of $|\overline{X}'_0\rangle$ and $|X'_h\rangle$ will have a larger overlap with $|\overline{X}_0\rangle$ than $|\overline{X}'_0\rangle$, violating the assumption.) It follows that

$$\begin{aligned}\langle X'_{h'}|X_h\rangle &= \langle X'_{h'}|U_I(h)|\overline{X}_0\rangle \\ &= \langle X'_{h'h^{-1}}|\overline{X}_0\rangle \sim \delta(h'h^{-1}).\end{aligned} \tag{14}$$

We can repeat the procedure and define a $|\overline{X}'_0\rangle$ for all v'. The overlap as a function of $w = v \cdot v'$ is denoted by $\eta(w)$ and will turns out to be the universal Isgur–Wise form factor.

$$\langle \overline{X}'_0|\overline{X}_0\rangle = \eta(w). \tag{15}$$

A rotation in isospace gives

$$\langle X'_h|X_h\rangle = \eta(w). \tag{16}$$

Combining Eqs. (14) and (16), we end up with

$$\langle X'_{h'}|X_h\rangle = \eta(w)\delta(h'h^{-1}). \tag{17}$$

This is the central result of this article, the overlap of any state in $\{|X_h\rangle : h \in \mathrm{SU}(2)\}$ with any state in $\{|X'_{h'}\rangle : h' \in \mathrm{SU}(2)\}$ can be expressed in terms of a single form factor $\eta(w)$. All remains to be done is to express the result in terms of the eigenstates of I^a and s^i_ℓ.

Following the notation of Ref. [1], $|I, a; s_\ell, m\rangle$ will denote a state with isospin I and "brown muck" spin s_ℓ, while a and m are the third components of the (iso)spin. Then $|0,0;0,0\rangle$ and $|1,a;1,m\rangle$ are the "brown mucks" of Λ_Q and $\Sigma_Q^{(*)}$ respectively. We can express $|0,0;0,0\rangle$ in terms of the X_h basis.

$$|0,0;0,0\rangle = \int dh |X_h\rangle, \tag{18}$$

and

$$\begin{aligned}\langle 0,0;0,0(v')|0,0;0,0(v)\rangle &= \int dh\, dh'\, \langle X'_{h'}|X_h\rangle \\ &= \eta(w) \int dh\, dh'\, \delta(h'h^{-1}) \\ &= \eta(w),\end{aligned} \tag{19}$$

justifying the notation of $\eta(w)$. On the other hand,

$$|1,a;1,m\rangle = \int dh D^{am}(h)|X_h\rangle, \tag{20}$$

and

$$\begin{aligned}\langle 1,a';1,m'(v')|1,a;1,m(v)\rangle &= \int dh\, dh'\, \langle X'_{h'}|D^{a'm'\dagger}(h')D^{am}(h)|X_h\rangle \\ &= \eta(w) \int dh\, dh'\, \delta(h'h^{-1}) D^{a'm'\dagger}(h') D^{am}(h) \\ &= \eta(w)\delta_{aa'}\delta_{mm'},\end{aligned} \tag{21}$$

by the orthonormality of the rotational matrices D^{am}'s. This is exactly (the second equality of) Eq. (16) of Ref. [1]. In terms of the full baryon states $|\Sigma_Q^*\rangle$, the result is

$$\langle \Sigma_c^*(v',\epsilon',s')|\bar{c}\Gamma b|\Sigma_b^*(v,\epsilon,s)\rangle = \frac{\eta(w)}{1+w}[(1+w)g_{\mu\nu} - v_\nu v'_\mu]\epsilon'^{*\nu}\epsilon^\mu \bar{u}_c \Gamma u_b, \tag{22}$$

where ϵ and ϵ' are the polarization vectors and s and s' are the heavy quark spins[2]. And when compared to Eq. (2), the main result of Ref. [1] is recovered.

---

[2] As noted in Ref. [7], the "east coast" metric $(-,+,+,+)$ is used.

$$\zeta_1(w) = -(1+w)\zeta_2(w) = \eta(w). \tag{23}$$

Since the proof above is quite complicated, let's consider an simple but problematic alternative proof which may help to bring out the essence of the correct proof above. Note that

$$|1, a; 1, m\rangle = X_0^{ma}|0, 0; 0, 0\rangle, \tag{24}$$

and hence

$$\begin{aligned}\langle 1, a'; 1, m'(v')|1, a; 1, m(v)\rangle &= \langle 0, 0; 0, 0(v)|X'^{m'a'\dagger}_0 X_0^{ma}|0, 0; 0, 0(v)\rangle \\ &= \langle 0, 0; 0, 0(v')|0, 0; 0, 0(v)\rangle, \end{aligned} \tag{25}$$

if one sloppily identifies the operators X_0 and X'_0. But such sloppiness is problematic. The operators X_0^{ma} carries a spatial index m, and the operator may get non-trivially transformed when boosted from the v frame to the v' frame. The isospin operator I^a, on the other hand, is manifestly independent of Lorentz frames. That is why the $|X_h\rangle$ basis is used: for these states, a rotation in the real space is equivalent to a rotation in the isospace. So, after identifying *one* state with velocity v with *one* with velocity v' (through the criterion of maximal overlap), one can establish a one-one correspondence between the two sets of states just by isorotations, which are frame-independent operations. These one-one corresponded states all have the same overlap, and that is the Isgur–Wise form factor.

It must be emphasized that this study does *not* question the computational correctness of Ref. [7]. While the authors of Ref. [7] try to obtain constraints on the form factor through unitarity, the present work depends mainly on the SU(4) symmetry structure of baryons in the large N_c limit. Since this SU(4) spin-flavor symmetry is completely model independent, and all existing approaches to large N_c baryons (chiral soliton, Hartree–Fock, etc.) exhibit this symmetry, our result is truly model independent. In fact, this universality of baryon Isgur–Wise form factor should hold in any formalism exhibiting this spin-flavor symmetry, no matter it is large N_c motivated or not. One such example is the constituent quark model developed in Ref. [11], which is not directly related to $1/N_c$ expansion but embodies the

same symmetry. In fact, the universality of baryon Isgur–Wise form factor made its first appearance in their work, which chronologically precedes Ref. [1].

In conclusion, it is found that the large N_c universality of baryon Isgur–Wise form factor discussed in Ref. [1] is solely the consequence of the SU(4) spin-flavor symmetry and is independent of any dynamical assumptions. This universality can be put to experimental test in the future, when more data are available on $\eta(w)$ (from $\Lambda_b \to \Lambda_c$ decays) and the $\zeta(w)$'s (from $\Omega_b \to \Omega_c$ decays). The deviation from universality is a measure of the (in)applicability of the large N_c expansion for heavy baryons.

ACKNOWLEDGMENTS

I am grateful to Tung–Mow Yan for discussions. This work is supported in part by the National Science Foundation.

REFERENCES

[1] C.K. Chow, Phys. Rev. **D51** 1224 (1995).

[2] N. Isgur and M.B. Wise, in "*B* Decay," ed. S. Stone.

[3] N. Isgur and M.B. Wise, Nucl. Phys. **B348** 276 (1991).

[4] H. Georgi, Nucl. Phys. **B348** 293 (1991).

[5] T. Mannel, W. Roberts and Z. Ryzak, Nucl. Phys. **B355** 38 (1991).

[6] F. Hussain, J.G. Korner, M. Kramer and G. Thompson, Z. Phy **C51** 321 (1991).

[7] D.E. Brahm and J. Walden, hep-ph 9511309 (1995).

[8] E. Jenkins, A.V. Manohar and M.B. Wise, Nucl. Phys. **B396** 38 (1993).

[9] R. Dashen, E. Jenkins and A.V. Manohar, Phys. Rev. **D49** 4713 (1994).

[10] C.K. Chow, Phys. Rev. **D51** 3587 (1995).

[11] J.G. Korner, M. Kramer and D. Pirjol, Prog. Part. Nucl. Phys. **33** 787 (1994).